

Quantum Dynamical Algebra $SU(1,1)$ in One-Dimensional Exactly Solvable Potentials

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In this paper, we establish the underlying quantum dynamical algebra $SU(1,1)$ for some one-dimensional exactly solvable potentials by using the shift operators method. The connection between $SU(1,1)$ algebra and the radial Hamiltonian problems is also discussed.

KEY WORDS: quantum dynamical algebra; exactly solvable potentials.

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1. INTRODUCTION

Exactly solvable models are very important in physics. They are important not just from a theoretical point of view but also from the experimentalist's perspective because in such cases theoretical results and experimental results can be compared without ambiguity. Exactly solvable potentials especially including one-dimensional or spherically symmetric ones have extensively played the indispensable roles in condensed matter, biophysics, nuclear physics, quantum optics and solid-state physics. Those familiar potentials embrace, for instance, the typical harmonic oscillator potential, conventional Coulomb potential, one dimensional Morse potential (1929), the Rosen–Morse potential (1932), the Pöschl–Teller potential (Pöschl and Teller, 1933; Grosche, 1989), the Hulthén potential, the Kratzer's molecular potential, and the famous Yukawa potential, etc. (Flügge, 1974). Thereinto, the Morse potential and Kratzer's molecular potential are utilized to describe the anharmonicity and bond dissociation of diatomic molecules. Another noticeable potential with short-range properties is the Pöschl–Teller potential, of which the generalized coherent states (Crawford and Vrsay, 1998), nonlinear properties (Chen *et al.*, 1998; Quesne, 1999), and

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supersymmetric extension (Díaz *et al.*, 1999) have been widely studied. Additionally, in supersymmetric (SUSY) quantum mechanics, the shaped invariant potentials have also been mooted (Gendenstein, 1983; Dutt *et al.*, 1986; Cooper *et al.*, 1988; Khare and Sukhatme, 1988; Dabrowska *et al.*, 1988) In particular, a large class of such potentials is the Natanzon class (Natanzon, 1979; Fukui and Aizawa, 1993; Dutt *et al.*, 1993; Roychoudhury *et al.*, 2001; Znojil *et al.*, 2001).

The preference to deal with those potentials in modern quantum mechanics adopts the abstract formulation and stresses the special nature of wave mechanics. However, the machinery of wave mechanics such as choice of coordinate system, separation of variables, boundary conditions, single-valuedness can obscure the underlying quantum mechanical principles and complicate the analysis. In this way, the operator methods which mainly consist of noncommutative algebra and the shift operator factorization to some extent can avoid these flaws. Algebraic methods (Kamran and Olver, 1990; Celeghini *et al.*, 1985) exploring the underlying Lie symmetry and its associated algebra have been widely used to study many of these exactly solvable potentials, for instance, Darboux transformation, Infeld–Hull transformation (Infeld and Hull, 1951; Stahlhofen and Bleuler, 1989), Mielnick factorization (1984), SUSY quantum mechanics, inverse scattering theory (Hoppe, 1992), and intertwining technique (Díaz *et al.*, 1999). Operators methods with shift operators (i.e., raising and lowering operators) for the Hamiltonian of exactly solvable potentials have been presented in De Lange and Raab (1991). In 1993, nonlinear deformations of $SU(2)$ and $SU(1,1)$ algebras (NLDA) with two deforming functions $g(J_0)$ and $f(J_0)$ are introduced by Delbecq and Quesne (1993). Subsequently, in 1998 Chen *et al.* (1998) applied successfully the nonlinear deformation algebra to a physical system with Pöschl–Teller potential, and furthermore they tuned the NLDA naturally to a linear algebra $SU(1,1)$ by readjusting the generators (Chen *et al.*, 1998; Quesne, 1999); in other words, the underlying quantum dynamical algebra (QDA) $SU(1,1)$ was revealed for the Pöschl–Teller potential, and one may note that the crucial step to establish the QDA is firstly to construct the shift operators for the potentials. In 2000, following the similar method a unified approach that emphasized on constructing the shift operators of exactly one-dimensional solvable potentials was provided (Ge *et al.*, 2000), but without pointing out the concomitant algebraic structures. The purpose of this paper is to go beyond Ge *et al.* (2000) and establish systematically the corresponding underlying quantum dynamical algebra for the potentials discussed. Interestingly and also surprisingly, these exactly solvable potentials possess the same simple algebra $SU(1,1)$.

The paper is organized as follows: In Sec. 2, we first briefly review the Pöschl–Teller potential problem discussed in Chen *et al.* (1998) and Quesne (1999), and then present a systematic method to construct the linear algebra $SU(1, 1)$ for one-dimensional exactly solvable potentials. In Sec. 3, quantum dynamical algebra $SU(1, 1)$ is established for some one-dimensional exactly solvable potentials. In

Sec. 4, we discuss the radial Hamiltonian problems, which in some sense can be viewed as one-dimensional problems. Conclusion and discussion are made in the last section.

2. QUANTUM DYNAMIC ALGEBRA OF ONE-DIMENSIONAL POTENTIAL

2.1. Brief Review

The nonlinear deformation algebra (NLDA) as noted in Delbecq and Quesne (1993) generated by three operators $b_0 = b_0^\dagger$, and $b_- = (b_+)^{\dagger}$ satisfying

$$[b_0, b_-] = -b_-g(b_0), \quad [b_0, b_+] = g(b_0)b_+, \quad [b_-, b_+] = f(b_0). \quad (1)$$

which can be realized for the Pöschl–Teller (PT) potential pointed out by Chen *et al.* (1998). Further, such NLDA of PT potential can be transformed into a $SU(1,1)$ algebra demonstrated in Chen *et al.* (1998) and Quesne (1999). The results are briefly reviewed as follows.

The Hamiltonian is $H = p^2/2m + V(x)$ with $V(x) = V_0/\cos^2(kx)$, and $V_0 = \varepsilon v(v - 1)$ accompanying by $\varepsilon = \hbar^2 k^2/2m$. After defining the generalized coordinate X and momentum P ,

$$X = \sin(kx), \quad P = \frac{1}{2}k\{\cos(kx), p\},$$

it found out the shift operator in Eq. (1)

$$\begin{aligned} b_0 &= H, \\ b_- &= \frac{1}{2\varepsilon} \left[X \left(\varepsilon + 2\sqrt{\varepsilon H} \right) + \frac{i\hbar}{m} P \right], \\ b_+ &= -\frac{1}{2\varepsilon} \left[X \left(\varepsilon - 2\sqrt{\varepsilon H} \right) + \frac{i\hbar}{m} P \right] \frac{\varepsilon + \sqrt{\varepsilon H}}{\sqrt{\varepsilon H}}. \end{aligned} \quad (2)$$

and

$$\begin{aligned} g(H) &= -\varepsilon + 2\sqrt{\varepsilon H}, \\ f(H) &= 1 + 2\sqrt{H/\varepsilon} + \frac{v(v - 1)}{\sqrt{H/\varepsilon} (\sqrt{H/\varepsilon} - 1)} \end{aligned}$$

Thus it realized the NLDA. Further, through redefining the three generator elements, that is

$$J_0 = \sqrt{H/\varepsilon}, \quad J_- = \left(\frac{\sqrt{H/\varepsilon}}{\sqrt{H/\varepsilon} + 1} \right)^{1/2}, \quad J_+ = J_-^\dagger$$

they satisfied the $SU(1, 1)$ algebra

$$[J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = -2J_0.$$

Therefore, it turned the nonlinear algebra into a linear dynamical algebra for such special PT potential that possesses a closed operator sets $\{H, X, P\}$. Such procedure also gives a suggestion that for an arbitrary one-dimensional potential, if finding its closed operator sets $\{H, X, P\}$, we can immediately construct the shift operators and then tune them into a linear algebra. The method to find the closed operator sets has been discussed in Ge *et al.* (2000), and in the next subsection we emphasize the method to tune them into a linear algebra.

2.2. The Main Method

For a given quantum system with Hamiltonian H , if there are generalized coordinate operator Q and generalized momentum operator P satisfying the following commutation relations as described in Ge *et al.* (2000)

$$[H, Q] = Q\Theta_1(H) + P\Pi_1(H),$$

$$[H, P] = Q\Theta_2(H) + P\Pi_2(H),$$

where $\Theta_i(H)$ and $\Pi_i(H)$ are functions about operator H , it is then able to find out the shift operators for the energy eigenstates by using the matrix-diagonalizing technique. That is, given

$$[H, (Q, P)] = (Q, P) \begin{pmatrix} \Theta_1 & \Theta_2 \\ \Pi_1 & \Pi_2 \end{pmatrix} \quad (3)$$

diagonalizing the matrix in the right obtain a transformation matrix S and leaves a diagonalized matrix Λ . Thus the Eq. (3) can be written as

$$[H, (Q, P)] = (Q, P)S\Lambda S^{-1}$$

with

$$\Lambda = \begin{pmatrix} -\Omega_1(H) & 0 \\ 0 & \Omega_2(H) \end{pmatrix}.$$

We can define the shift operators S_1 and S_2 to be

$$(S_1, S_2) = (Q, P)S$$

which satisfy

$$H(S_1, S_2) = (S_1, S_2) \begin{pmatrix} H - \Omega_1(H) & 0 \\ 0 & H + \Omega_2(H) \end{pmatrix} \quad (4)$$

Further, for a real function $F(H)$, holomorphic in the neighborhood of zero, it is easy to derive the following identity

$$F(H)(S_1, S_2) = (S_1, S_2) \begin{pmatrix} F(H - \Omega_1) & 0 \\ 0 & F(H + \Omega_2) \end{pmatrix} \tag{5}$$

From the Eqs. (4) and (5) we can always find such operator function $b_0(H)$ that satisfy

$$b_0(H)(S_1, S_2) = (S_1, S_2) \begin{pmatrix} b_0(H) - 1 & 0 \\ 0 & b_0(H) + 1 \end{pmatrix} \tag{6}$$

which have a more familiar form as

$$[b_0, S_1] = -S_1, \quad [b_0, S_2] = S_2. \tag{7}$$

Such commutation relation implies a freedom that

$$b_0(H) \rightarrow b_0(H) + \xi_0 \tag{8}$$

with ξ_0 a constant keeps the Eq. (7) unchangeable. This freedom would be explored to select out the Lie algebraic element J_0 . Additionally, the identity (5) evolved into

$$G(b_0)(S_1, S_2) = (S_1, S_2) \begin{pmatrix} G(b_0 - 1) & 0 \\ 0 & G(b_0 + 1) \end{pmatrix} \tag{9}$$

One noticeable point is that there is a freedom of multiplying a function $\xi_i(b_0)$ on S_1 and S_2 , that is

$$\begin{aligned} S_1 &\rightarrow \xi_1(b_0)S_1 \quad \text{or} \quad S_1\xi_1(b_0), \\ S_2 &\rightarrow \xi_2(b_0)S_2 \quad \text{or} \quad S_2\xi_2(b_0) \end{aligned} \tag{10}$$

which still satisfy the Eqs. (4) and (5). This freedom can be used to guarantee finding the Lie algebraic elements J_+ and J_- from S_1 and S_2 .

As for the constructed shift operators S_1 and S_2 , generally they are not Hermite conjugate with each other but satisfy

$$S_1^\dagger = S_2\eta(b_0). \tag{11}$$

which makes it easy to substitute

$$b = S_1, \quad b^\dagger = S_1^\dagger = S_2\eta(b_0) \tag{12}$$

with $b^\dagger = (b)^\dagger$. For the purpose of constructing an algebra structure, we can define

$$J_0 = b_0 + \xi_0, \quad J_+ = b^\dagger\xi(b_0), \quad J_- = \xi(b_0)b \tag{13}$$

which have naturally satisfied

$$[J_0, J_+] = J_+, \quad [J_0, J_-] = -J_-, \quad J_+ = J_-^\dagger,$$

and $\xi_0, \xi(b_0)$ need finally determined by the case

$$[J_+, J_-] = \pm 2J_0, \tag{14}$$

where ‘+’ corresponds to $SU(2)$ while ‘-’ to $SU(1, 1)$.

The commutation relation from the Eqs. (9) and (11) is reduced into

$$\begin{aligned} [J_+, J_-] &= b^\dagger \xi(b_0)^2 b - \xi(b_0) b b^\dagger \xi(b_0) \\ &= S_2 S_1 \eta(b_0 - 1) \xi(b_0 - 1)^2 - S_1 S_2 \eta(b_0) \xi(b_0)^2 \end{aligned}$$

which would be treated in three cases.

- The $\eta(b_0) = \text{Constant}$, but $[S_1, S_2] = f(b_0) \neq \text{Constant}$ case. We can simply put, $\xi(b_0) = \xi(b_0 - 1) = \text{Constant}$, and thus

$$[J_+, J_-] = [S_2, S_1] \eta \xi = f(b_0) \eta \xi = \pm 2J_0 \tag{15}$$

- The $\eta(b_0) = \text{Constant}$, but $[S_1, S_2] = \text{Constant}$ case. We must choose $\xi(b_0) = f(b_0) \neq \text{Constant}$ to satisfy

$$[J_+, J_-] = [S_2, S_1 f(b_0 - 1) - S_1, S_2 f(b_0)] \xi^2 = \pm 2J_0 \tag{16}$$

- The $\eta(b_0) \neq \text{Constant}$ case. We can choose $\xi(b_0) \neq \text{Constant}$ to satisfy

$$\eta(b_0 - 1) \xi(b_0 - 1)^2 = \eta(b_0) \xi(b_0)^2 = g(b_0), \tag{17}$$

and thus

$$[J_+, J_-] = [S_2, S_1] g(b_0) = \pm 2J_0. \tag{18}$$

In this way, the Eqs. (15)–(17), and (18) can be finally used to derive out the algebra elements J_0, J_\pm .

One simple example concerns the one-dimensional harmonic oscillator with the Hamiltonian, $H = \frac{1}{2}(x^2 + p^2)$, where x is the coordinate operator and $p = -i(d/dx)$ is the momentum operator with commutator given by $[x, p] = i$. For the commutation relations $[H, x] = -ip$ and $[H, p] = ix$, they as noted in Ge *et al.* (2000) can be succinctly written as

$$[H, (x, p)] = (x, p) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Finding out its shift operators a and a^\dagger , it gives

$$[H, (a, a^\dagger)] = (a, a^\dagger) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

in which $a \equiv (1/\sqrt{2})(x + ip)$ and $a^\dagger \equiv (1/\sqrt{2})(x - ip)$ with

$$a a^\dagger = H + \frac{1}{2}, \quad [a, a^\dagger] = 1. \tag{19}$$

Thus, from Eqs. (6) and (12) we know

$$J_0 = H, \quad b = a, \quad b^\dagger = a^\dagger. \tag{20}$$

It belongs to the second case. Inserting Eqs. (19) and (20) into Eq. (16), it deduces

$$\left(H - \frac{1}{2}\right)\xi(H - 1)^2 - \left(H + \frac{1}{2}\right)\xi(H)^2 = -2H.$$

Obviously, by choosing $\xi(H)^2 = H + 1/2$, the above equation would hold. Therefore, the one-dimensional harmonic oscillator has a $SU(1,1)$ algebra with

$$J_0 = H, \quad J_+ = a^\dagger\sqrt{H + 1/2}, \quad J_- = \sqrt{H + 1/2}a.$$

3. QUANTUM DYNAMICAL ALGEBRA FOR MORE POTENTIALS

As pointed out in Ge *et al.* (2000), the shift operators of a class of solvable one-dimension potentials can be found by using the matrix-diagonalizing technique. In the following, we will continue to explore such technique to display that all these potentials possess either a $SU(1,1)$ or a $SU(2)$ algebra. To this end, we begin with the following Hamiltonian:

$$H = X(x)\frac{d^2}{dx^2} + V(x),$$

where $X(x)$ is an arbitrary function of position and $V(x)$ is an arbitrary potential. Define

$$P(x, p) = Y(x)\frac{d}{dx} + Z(x);$$

with arbitrary functions $Y(x)$ and $Z(x)$ to be determined. The following commutation relations hold

$$\begin{aligned} [H, P(x)] &= Q(x)(\beta H + 1) + \alpha P + \gamma H, \\ [H, Q(x)] &= 2\lambda P + \nu Q(x) + \tau. \end{aligned} \tag{21}$$

with

$$Q(x) = \frac{1}{1 + \beta V(x)} [X(x)Z''(x) - \gamma V(x) - \alpha Z(x) - Y(x)V'(x)]. \tag{22}$$

Concomitantly, $X(x)$, $V(x)$, $Y(x)$ and $Z(x)$ satisfy

$$\begin{aligned} X(x)[Y''(x) + 2Z'(x)] &= \alpha Y(x), \\ 2X(x)Y'(x) = X'(x)Y(x) &= [\beta Q(x) + \gamma]X(x), \\ X(x)Q'(x) &= \lambda Y(x), \\ -2\lambda Z(x) + X(x)Q''(x) &= \nu Q(x) + \tau. \end{aligned} \tag{23}$$

In the construction, we should bear in mind that $Y(x)$, $Z(x)$, $Q(x)$, and α , β , γ , λ , ν , τ , are all determined by $X(x)$ and $V(x)$ when we construct the shift operators for a given Hamiltonian. Conversely, if we proceed with a systematic search for exactly solvable potentials based on this method, we can have as many degrees of freedom as there are free parameters and functions, namely, $X(x)$, $Y(x)$, $Z(x)$, $Q(x)$, and α , β , γ , λ , ν , τ .

3.1. The $X(x) = -1$, $Y(x) = i$ Case

By substituting into Eq. (23), we solved

$$Q(x) = -i\lambda x + c_i, \quad Z(x) = -\frac{i}{2}\alpha x + c_2, \quad (24)$$

and

$$V(x) = \frac{1}{2} \left(\lambda + \frac{\alpha^2}{2} \right) x^2 - (c_1 + \alpha c_2) i x + c_3, \quad (25)$$

with $\beta = 0$, $\gamma = 0$, $\nu = -\alpha$ and $\tau = -c_1\nu - 2\lambda c_2$. This is the harmonic oscillator when the coefficient $\lambda + \alpha^2/2 > 0$. For the requirement of Hermitian operators H and P , it requires the parameters α , c_1 being imaginary numbers while γ , c_2 , c_3 real numbers.

The closed operator set $\{H, \tilde{Q}, \tilde{P}\}$ satisfy the commutation relations

$$[H, \tilde{Q}] = -\alpha \tilde{Q} + 2\lambda \tilde{P}, \quad [H, \tilde{P}] = \tilde{Q} + \alpha \tilde{P}. \quad (26)$$

or more succinctly as

$$[H, (\tilde{Q}, \tilde{P})] = (\tilde{Q}, \tilde{P}) \begin{pmatrix} -\alpha & 1 \\ 2\lambda & \alpha \end{pmatrix} \quad (27)$$

in which

$$\begin{aligned} \tilde{Q} &= Q + \alpha \frac{2\lambda c_2 + \nu c_1}{\alpha^2 + 2\lambda} = -i\lambda x + c_1 + \alpha \frac{2\lambda c_2 + \nu c_1}{\alpha^2 + 2\lambda}, \\ \tilde{P} &= P - \frac{2\lambda c_2 + \nu c_1}{\alpha^2 + 2\lambda} = i \frac{d}{dx} - \frac{i}{2}\alpha x + \frac{\alpha^2 c_2 - \nu c_1}{\alpha^2 + 2\lambda}. \end{aligned} \quad (28)$$

It can be easily diagonalized to give the shift operators S_1 and S_2 which satisfied the commutation relation

$$[H, (S_1, S_2)] = (S_1, S_2) \begin{pmatrix} -\sqrt{\alpha^2 + 2\lambda} & 0 \\ 0 & \sqrt{\alpha^2 + 2\lambda} \end{pmatrix} \quad (29)$$

with

$$S_1 = \frac{1}{\alpha - \sqrt{\alpha^2 + 2\lambda}} \tilde{Q} + \tilde{P},$$

$$S_2 = \frac{1}{\alpha + \sqrt{\alpha^2 + 2\lambda}} \tilde{Q} + \tilde{P} \tag{30}$$

which satisfy the relation $(S_1)^\dagger = S_2$ and

$$[S_1, S_2] = \sqrt{\alpha^2 + 2\lambda}. \tag{31}$$

So it belongs to the second case. Thus, we can obtain a $SU(1, 1)$ algebra according to the second treatment after defining

$$J_0 = H/\sqrt{\alpha^2 + 2\lambda}, \tag{32}$$

$$J_- = \frac{\sqrt{J_0 + 1/2}}{(\alpha^2 + 2\lambda)^{1/4}} b, \quad J_+ = b^\dagger \frac{\sqrt{J_0 + 1/2}}{(\alpha^2 + 2\lambda)^{1/4}}. \tag{33}$$

3.2. The $X(x) = -1, Y(x) = x$ Case

By substituting into Eq. (23), we solved

$$Q(x) = -\frac{\lambda}{2}x^2 + c_1, \quad Z(x) = -\frac{\alpha}{4}x^2 + c_2, \tag{34}$$

and

$$V(x) = \frac{1}{16}(\alpha^2 + 2\lambda)x^2 + \frac{c_3}{x^2} + \frac{1}{2} \left(\frac{\alpha}{2} - \alpha c_2 - c_1 \right), \tag{35}$$

with $\beta = 0, \gamma = 2, \nu = -\alpha$ and $\tau = (1 - 2c_2)\lambda + \alpha c_1$. this gives us the radial harmonic oscillator potential.

The closed operator set $\{H, \tilde{Q}, \tilde{P}\}$ satisfy the commutation relations

$$[H, (\tilde{Q}, \tilde{P})] = (\tilde{Q}, \tilde{P}) \begin{pmatrix} -\alpha & 1 \\ 2\lambda & \alpha \end{pmatrix} \tag{36}$$

in which

$$\begin{aligned} \tilde{Q} &= Q + \frac{4\lambda}{\alpha^2 + 2\lambda} H - \frac{\alpha}{\alpha^2 + 2\lambda} (\lambda + \alpha c_1 - 2\lambda c_2) \\ &= -\frac{\lambda}{2}x^2 + c_1 + \frac{4\lambda}{\alpha^2 + 2\lambda} H - \frac{\alpha}{\alpha^2 + 2\lambda} (\lambda + \alpha c_1 - 2\lambda c_2), \\ \tilde{P} &= x \frac{d}{dx} - \frac{\alpha}{4}x^2 + c_2 + \frac{2\alpha}{\alpha^2 + 2\lambda} H + \frac{1}{\alpha^2 + 2\lambda} (\lambda + \alpha c_1 - 2\lambda c_2). \end{aligned}$$

The shift operators S_1 and S_2 which satisfied the commutation relation

$$[H, (S_1, S_2)] = (S_1, S_2) \begin{pmatrix} -\sqrt{\alpha^2 + 2\lambda} & 0 \\ 0 & \sqrt{\alpha^2 + 2\lambda} \end{pmatrix}, \tag{37}$$

and

$$[S_1, S_2] = 16 \frac{\lambda}{\sqrt{\alpha^2 + 2\lambda}} \left[H - \frac{1}{2} \left(\frac{\alpha}{2} - \alpha c_2 - c_1 \right) \right],$$

with

$$S_1 = \tilde{Q} + (\alpha - \sqrt{\alpha^2 + 2\lambda}) \tilde{P},$$

$$S_2 = \tilde{Q} + (\alpha + \sqrt{\alpha^2 + 2\lambda}) \tilde{P}$$

The shift operator S_1 and S_2 have

$$\begin{aligned} S_1^\dagger &= \left(1 + \frac{\alpha^2 - \alpha\sqrt{\alpha^2 + 2\lambda}}{\lambda} \right) \tilde{Q} + (-\alpha + \sqrt{\alpha^2 + 2\lambda}) \tilde{P} \\ &= S_2 \left(1 + \frac{\alpha^2 - \alpha\sqrt{\alpha^2 + 2\lambda}}{\lambda} \right) \end{aligned}$$

Therefore we can define

$$\begin{aligned} b_0 &= \frac{H}{\sqrt{\alpha^2 + 2\lambda}}, \quad b = S_1, \\ b^\dagger &= S_2 \left(1 + \frac{\alpha^2 - \alpha\sqrt{\alpha^2 + 2\lambda}}{\lambda} \right). \end{aligned}$$

Thus it belongs to the first case. According to the first treatment, we can define again

$$J_0 = b_0 - \frac{1}{2\sqrt{\alpha^2 + 2\lambda}} \left(\frac{\alpha}{2} - \alpha c_2 - c_1 \right) \quad (38)$$

$$J_+ = \frac{1}{2\sqrt{2}} (\alpha^2 + \lambda - \alpha\sqrt{\alpha^2 + 2\lambda})^{-1/2} b^\dagger, \quad (39)$$

$$J_- = \frac{1}{2\sqrt{2}} (\alpha^2 + \lambda - \alpha\sqrt{\alpha^2 + 2\lambda})^{-1/2} b, \quad (40)$$

which satisfy the $SU(1,1)$ algebra

$$[J_+, J_-] = -2J_0.$$

3.3. The $X(x) = -1, Y(x) = ae^{cx} + be^{-cx}$ Case

By substituting into Eq. (23), we solved

$$Q(x) = -\frac{\lambda}{c} (ae^{-cx} - be^{-cx}) + c_1, \quad (41)$$

$$Z(x) = -\frac{\alpha + c^2}{2c}(ae^{cx} - be^{-cx}) + c_2, \tag{42}$$

and

$$V(x) = c_3(ae^{cx} + be^{-cx})^{-2} + \frac{(\alpha + c^2)^2 + c^4 + 2\lambda}{4c^2}, \tag{43}$$

with $\beta = -2c^2/\lambda$, $\gamma = 2c^2c_1/\lambda$, $\nu = -\alpha - 2c^2$, $\tau = -2\lambda c_2 - \nu c_1$, and $c_1 + \alpha c_2 = 0$. This gives us the second Pöschl–Teller potential.

The closed operator set $\{H, \tilde{Q}, \tilde{P}\}$ satisfy the commutation relations

$$[H, (\tilde{Q}, \tilde{P})] = (\tilde{Q}, \tilde{P}) \begin{pmatrix} -\alpha - 2c^2 & 1 - \frac{2c^2}{\lambda}H \\ 2\lambda & \alpha \end{pmatrix} \tag{44}$$

in which

$$\tilde{Q} = Q - c_1 = -\frac{\lambda}{c}(ae^{cx} - be^{-cx}) \tag{45}$$

$$\tilde{P} = P + \frac{c_1}{\alpha} - (ae^{cx} + be^{-cx})\frac{d}{dx} - \frac{\alpha + c^2}{2c}(ae^{cx} - be^{-cx}) \tag{46}$$

The shift operators S_1 and S_2 satisfy the commutation relation

$$[H, S_1] = S_1[-c^2 - \sqrt{(\alpha + c^2)^2 + 2\lambda - 4c^2H}], \tag{47}$$

$$[H, S_2] = S_2[-c^2 + \sqrt{(\alpha + c^2)^2 + 2\lambda - 4c^2H}]. \tag{48}$$

with

$$\begin{aligned} S_1 &= -\tilde{Q} \frac{1}{2\lambda} [\alpha + c^2 + \sqrt{(\alpha + c^2)^2 + 2\lambda - 4c^2H}] + \tilde{P}, \\ S_2 &= -\tilde{Q} \frac{1}{2\lambda} [\alpha + c^2 - \sqrt{(\alpha + c^2)^2 + 2\lambda - 4c^2H}] + \tilde{P}. \end{aligned} \tag{49}$$

and

$$\begin{aligned} [S_1, S_2] &= -8abc^2b_0, \\ S_2^\dagger &= S_1 \left[\frac{2c^2}{\sqrt{(\alpha + c^2)^2 + 2\lambda - 4c^2H}} - 1 \right]. \end{aligned}$$

From the Eq. (5), it gives

$$[\sqrt{(\alpha + c^2)^2 + 2\lambda - 4c^2H}, S_1] = 2c^2S_1, \tag{50}$$

$$[\sqrt{(\alpha + c^2)^2 + 2\lambda - 4c^2H}, S_2] = -2c^2S_2, \tag{51}$$

Therefore we can define

$$b_0 = \frac{1}{2c^2} \sqrt{(\alpha + c^2)^2 + 2\lambda - 4c^2H}, \tag{52}$$

$$b = S_2, \quad b^\dagger = S_1 \left[\frac{2c^2}{\sqrt{(\alpha + c^2)^2 + 2\lambda - 4c^2H}} - 1 \right].$$

It belongs to the third case and according to the third treatment, the Eqs. (17) and (18) becomes

$$\left(\frac{1}{b_0 - 1} - 1 \right) \xi(b_0 - 1)^2 = \left(\frac{1}{b_0} - 1 \right) \xi(b_0)^2, \tag{53}$$

$$[J_+, J_-] = [S_1, S_2] \eta(b_0) \xi^2(b_0) \tag{54}$$

There is a simplest choice for $\xi(b_0)$ to satisfy the Eq. (53), that is

$$\xi(b_0)^2 = \xi_1 \frac{b_0}{1 - b_0}, \tag{55}$$

where ξ_1 is also a undetermined constant. Insert it into Eq. (54), we can determine the two constants ξ_0 and ξ_1 as

$$\xi_0 = 0, \quad \xi_1 = \frac{1}{4abc^2} \tag{56}$$

Finally, we can realize our algebraic structure by defining again

$$J_0 = b_0, \quad J_- = \frac{1}{\sqrt{4abc^2}} \sqrt{\frac{b_0}{b_0 - 1}} b, \quad J_+ = \frac{1}{\sqrt{4abc^2}} b^\dagger \sqrt{\frac{b_0}{b_0 - 1}} \tag{57}$$

which satisfy the $SU(1, 1)$ algebra,

$$[J_0, J_-] = -J_-, \quad [J_0, J_+] = J_+, \quad [J_+, J_-] = -2J_0.$$

3.4. The $X(x) = -1, Y(x) = a \sin(kx) + b \cos kx$ Case

By substituting into Eq. (23), we solved

$$Q(x) = \frac{\lambda}{k} [a \cos(kx) - b \sin(kx)] + c_1, \tag{58}$$

$$Z(x) = \frac{\alpha - k^2}{2k} [a \cos kx - b \sin kx] + c_2, \tag{59}$$

and

$$V(x) = c_3(a \sin kx + b \cos kx)^{-2} - \frac{(\alpha - k^2)^2 + 2\lambda}{4k^2}, \tag{60}$$

with $\beta = 2k^2/\lambda$, $\gamma = -2k^2c_1/\lambda$, $v = -\alpha + 2k^2$, $\tau = c_2(\alpha - 2k^2) - 2\lambda c_1$, and $c_1 + \alpha c_2 = 0$. This gives us the first Pöschl–Teller potential.

The closed operator set $\{H, \tilde{Q}, \tilde{P}\}$ satisfy the commutation relations

$$[H, (\tilde{Q}, \tilde{P})] = (\tilde{Q}, \tilde{P}) \begin{pmatrix} 2k^2 - \alpha & 1 + \frac{2k^2}{\lambda} H \\ 2\lambda & \alpha \end{pmatrix} \tag{61}$$

in which

$$\tilde{Q} = Q - c_1 = \frac{\lambda}{k}(a \cos kx - b \sin kx) \tag{62}$$

$$\tilde{P} = P - c_2 = (a \sin kx + b \cos kx) \frac{d}{dx} + \frac{\alpha - k^2}{2k}(a \cos kx - b \sin kx) \tag{63}$$

The shift operators S_1 and S_2 which satisfied the commutation relation

$$[H, S_1] = S_1[k^2 - \sqrt{(\alpha - k^2)^2 + 2\lambda + 4k^2H}],$$

$$[H, S_2] = S_2[k^2 + \sqrt{(\alpha - k^2)^2 + 2\lambda + 4k^2H}].$$

with

$$S_1 = -\tilde{Q} \frac{1}{2\lambda} [\alpha - k^2 + \sqrt{(\alpha - k^2)^2 + 2\lambda + 4k^2H}] + \tilde{P},$$

$$S_2 = -\tilde{Q} \frac{1}{2\lambda} [\alpha - k^2 - \sqrt{(\alpha - k^2)^2 + 2\lambda + 4k^2H}] + \tilde{P}.$$

We can derive that

$$S_1^\dagger = -S_2 \left[1 + \frac{2k^2}{\sqrt{(\alpha - k^2)^2 + 2\lambda + 4k^2H}} \right]$$

$$[S_1, S_2] = -(a^2 + b^2)\sqrt{(\alpha - k^2)^2 + 2\lambda + 4k^2H}.$$

From the Eq. (5), it gives

$$[\sqrt{(\alpha - k^2)^2 + 2\lambda + 4k^2H}, S_1] = -2k^2 S_1, \tag{64}$$

$$[\sqrt{(\alpha - k^2)^2 + 2\lambda + 4k^2H}, S_2] = 2k^2 S_2, \tag{65}$$

Therefore it is easy to define

$$b_0 = \frac{1}{2k^2} \sqrt{(\alpha - k^2)^2 + 2\lambda + 4k^2H}, \tag{66}$$

$$b = S_1,$$

$$b^\dagger = S_1^\dagger = -S_2 \left[1 + \frac{2k^2}{\sqrt{(\alpha - k^2)^2 + 2\lambda + 4k^2H}} \right]. \tag{67}$$

It belongs to the third case, and according to the treatment the Eqs. (17) and (18) becomes

$$\left(1 + \frac{1}{b_0}\right) \xi(b_0)^2 = \left(1 + \frac{1}{b_0 - 1}\right) \xi(b_0 - 1)^2, \quad (68)$$

$$[J_+, J_-] = [S_1, S_2] \left(1 + \frac{1}{b_0}\right) \xi(b_0)^2. \quad (69)$$

The simplest choice for $\xi(b_0)$ satisfying the condition (68) is

$$\xi(b_0)^2 = \xi_1 \frac{b_0}{b_0 + 1}. \quad (70)$$

Taking this choice into the Eq. (69), we obtain

$$[J_+, J_-] = -2k^2(a^2 + b^2)\xi_1 b_0, \quad (71)$$

which at the case of the $SU(1, 1)$ requires

$$\xi_1 = \frac{1}{k^2(a^2 + b^2)} \quad (72)$$

Finally, we can realize our algebraic structure by defining again

$$J_0 = b_0, \quad J_- = \frac{1}{k\sqrt{a^2 + b^2}} \sqrt{\frac{b_0}{b_0 + 1}} b, \quad J_+ = \frac{1}{k\sqrt{a^2 + b^2}} b^\dagger \sqrt{\frac{b_0}{b_0 + 1}} \quad (73)$$

which satisfy the $SU(1, 1)$ algebra,

$$[J_0, J_-] = -J_-, \quad [J_0, J_+] = J_+, \quad [J_+, J_-] = -2J_0.$$

4. EXTENDED TO RADIAL POTENTIALS

At case of dealing with the spherically symmetric potentials, their wave-functions can be divided into angular parts $Y_{lm}(\theta, \varphi)$ and radial parts $R_l(r)$. For the radial part $R_l(r)$, that it only depends on one variable is similar to the one-dimensional problem, so that we can try to use the methods introduced above to discuss it more or less.

Moreover, the radial part of the Schrödinger equation is generally written as

$$\left[\frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2} (E - V(r)) - \frac{l(l+1)}{r^2} \right] \chi_l(r) = 0,$$

where $\chi(r) = R_l(r)r$. We now try to discuss the radial Coulomb problem or more its extension of Kratzer's molecular potential by using the above method. Their

Schrödinger equations are

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2Z}{r} + \frac{Z^2}{n^2}\right)\psi_{n,l} = 0$$

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1) + Da^2}{r^2} - \frac{2Da}{r} + \frac{D^2a^2}{(n+1/2 + \sqrt{(l+1/2)^2 + \gamma^2})^2}\right]\psi_{n,l} = 0$$

which if making the substitutions $\rho = (Z/n)r$ or $\rho = \frac{Da}{n+1/2 + \sqrt{(l+1/2)^2 + \gamma^2}}r$ can be transformed into the easily disposed forms

$$\left[-\frac{d^2}{d\rho^2} + \frac{l(l+1)}{\rho^2} + 1 - \frac{2n}{\rho}\right]\psi_{n,l} = 0,$$

$$\left[-\frac{d^2}{d\rho^2} + \frac{l(l+1) + Da^2}{\rho^2} + 1 - 2\frac{n + \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 + \gamma^2}}{\rho}\right]\psi_{n,l} = 0.$$

Further, such form has two extended deformations which by virtue of the method discussed above are related to the energy quantum number n and the orbital quantum number l , respectively. One of the deformations is

$$\left[-\rho \frac{d^2}{d\rho^2} + \frac{l(l+1)}{\rho} + \rho\right]\psi_{n,l} = 2n\psi_{n,l},$$

$$\left[-\rho \frac{d^2}{d\rho^2} + \frac{l(l+1) + Da^2}{\rho^2} + \rho\right]\psi_{n,l} = 2\left[n + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 + \gamma^2}\right]\psi_{n,l}.$$

(74)

And the other one is

$$\left[-\rho^2 \frac{d^2}{d\rho^2} + \rho^2 - 2n\rho + l^2\right]\psi_{n,l} = -l\psi_{n,l},$$

$$\left[-\rho^2 \frac{d^2}{d\rho^2} + \rho^2 - 2\left(n + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 + \gamma^2}\right)\rho + l^2 + Da^2\right]\psi_{n,l} = -l\psi_{n,l}.$$

(75)

The left hand of the above deformed Schrödinger equations may well be called pseudo-Hamiltonian, which are no longer self-adjoint.

4.1. The $X(x) = -x, Y(x) = x$ Case

By substituting into Eq. (23), we solved

$$\begin{aligned} Q(x) &= -\lambda x + c_1, \\ Z(x) &= -\frac{\alpha}{2}x + c_2, \end{aligned}$$

and

$$V(x) = \frac{1}{2} \left(\lambda + \frac{\alpha^2}{2} \right) x + \frac{c_3}{x} - (c_1 + \alpha c_2),$$

with $\beta = 0, \gamma = 1, \nu = -\alpha, \tau = -2\lambda c_2 - \nu c_1$, and $c_1 + \alpha c_2 = 0$. This is the case of the first deformation [see Eq. (74)].

The closed operator set $\{H, \tilde{Q}, \tilde{P}\}$ satisfy the commutation relations

$$[H, (\tilde{Q}, \tilde{P})] = (\tilde{Q}, \tilde{P}) \begin{pmatrix} -\alpha & 1 \\ 2\lambda & \alpha \end{pmatrix}$$

in which

$$\begin{aligned} \tilde{Q} &= Q + \frac{2\lambda H - \alpha^2 c_1 + 2\lambda c_2 \alpha}{\alpha^2 + 2\lambda} = -\lambda x + \frac{2\lambda}{\alpha^2 + 2\lambda} H + 2\lambda \frac{c_1 + c_2 \alpha}{\alpha^2 + 2\lambda} \\ \tilde{P} &= P + \frac{\alpha H + \alpha c_1 - 2\lambda c_2}{2\lambda + \alpha^2} = x \frac{d}{dx} - \frac{\alpha}{2} x + \frac{\alpha}{\alpha^2 + 2\lambda} H + \alpha \frac{c_1 + \alpha c_2}{\alpha^2 + 2\lambda} \end{aligned}$$

The shift operators S_1 and S_2 satisfy the commutation relation

$$[H, (S_1, S_2)] = (S_1, S_2) \begin{pmatrix} -\sqrt{\alpha^2 + 2\lambda} & 0 \\ 0 & \sqrt{\alpha^2 + 2\lambda} \end{pmatrix}.$$

with

$$\begin{aligned} S_1 &= -\tilde{Q} \frac{\alpha + \sqrt{\alpha^2 + 2\lambda}}{2\lambda} + \tilde{P}, \\ S_2 &= -\tilde{Q} \frac{\alpha - \sqrt{\alpha^2 + 2\lambda}}{2\lambda} + \tilde{P}. \end{aligned}$$

which have

$$\begin{aligned} S_1^\dagger &= -S_2 - 1 + \frac{1}{\sqrt{\alpha^2 + 2\lambda}} \frac{d}{dx}, \\ [S_1, S_2] &= -\frac{2}{\sqrt{\alpha^2 + 2\lambda}} (H + c_1 + \alpha c_2). \end{aligned}$$

Here if we still define

$$b_0 = \frac{H}{\sqrt{\alpha^2 + 2\lambda}}, \quad b_- = S_1, \quad b_+ = S_2.$$

they can satisfy

$$[b_0, b] = -b, \quad [b_0, b^\dagger] = b^\dagger$$

but $b_0^\dagger \neq b_0$ and $b_-^\dagger \neq b_+$. So for such pseudo-Hamiltonian we can realize its algebraic structure by defining again

$$J_0 = b_0 + \frac{c_1 + \alpha c_2}{\sqrt{\alpha^2 + 2\lambda}}, \quad J_+ = b^\dagger, \quad J_- = b. \tag{76}$$

which satisfy $[J_+, J_-] = -2J_0$ but at a price of $J_0^\dagger \neq J_0$ and $J_-^\dagger \neq J_+$. So it does not mean we have constructed a $SU(1, 1)$ algebra.

4.2. The $X(x) = -x^2, Y(x) = 1$ Case

By substituting into Eq. (23), we solved

$$Q(x) = \frac{\lambda}{x} + \frac{\tau}{2}, \tag{77}$$

$$Z(x) = 0, \tag{78}$$

and

$$V(x) = -c_3x^2 + \frac{\gamma\lambda}{2}x + \frac{\lambda}{2}$$

with $\beta = -2/\lambda, 2\gamma = -\beta\tau, \nu = -2$ and $\alpha = 0$. This is simply the case of the second deformation [see Eq. (75)].

The closed operator set $\{H, \tilde{Q}, \tilde{P}\}$ satisfy the commutation relation

$$[H, \tilde{Q}, \tilde{P}] = (\tilde{Q}, \tilde{P}) \begin{pmatrix} -2 & -\frac{2}{\lambda}H + 1 \\ 2\lambda & 0 \end{pmatrix}$$

in which

$$\tilde{Q} = \frac{\lambda}{x} + \frac{\tau}{2} + \frac{\tau}{\lambda - 2H}H,$$

$$\tilde{P} = \frac{d}{dx} + \frac{\tau}{2(\lambda - 2H)}.$$

$$[H, (S_1, S_2)] = (S_1, S_2) \begin{pmatrix} -1 + \sqrt{1 + 2\lambda - 4H} & 0 \\ 0 & -1 - \sqrt{1 + 2\lambda - 4H} \end{pmatrix}.$$

$$[\sqrt{1 + 2\lambda - 4H}, (S_1, S_2)] = (S_1, S_2) \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}.$$

with

$$\begin{aligned} S_1 &= \tilde{Q}(\sqrt{1+2\lambda-4H}-1) + 2\lambda\tilde{P}, \\ S_2 &= -\tilde{Q}(\sqrt{1+2\lambda-4H}+1) + 2\lambda\tilde{P}. \end{aligned}$$

So we can define

$$J_0 = \frac{1}{2}\sqrt{1+2\lambda-4H}, \quad J_- = S_1, \quad J_+ = S_2$$

which can satisfy

$$[J_0, J_-] = -J_-, \quad [J_0, J_+] = J_+.$$

5. DISCUSSION AND CONCLUSION

We mainly explore the linear algebraic structure like $SU(2)$ or $SU(1,1)$ of the shift operators for some one-dimensional exactly solvable potentials in this paper. During such process, a set of method based on original diagonalizing technique is presented to construct those suitable operator elements, J_0, J_{\pm} that satisfy $SU(2)$ or $SU(1,1)$ algebra. A quick glance at the energy levels of the various cases shows that new-defined element operator J_0 has the same eigenvalues as that of the Hamiltonian of the harmonic oscillator. This fact also confirms to some extent that the local behavior of the most solvable potentials reduces to the harmonic oscillator (Wehrhahn, 1992). With $J_- \psi_0 = 0$, we can get the ground state ψ_0 , and $J_+ \psi_n = C \psi_{n+1}$ to get the whole spectrum. At the same time, $J_0 \psi_n = n \psi_n$ would indirectly give out the eigenvalues of Hamiltonian H .

Since the importance of spherically symmetric potentials in quantum mechanics, in Sec. 4 we discuss the deformed radial Hamiltonian of the hydrogen atom and Kratzer's molecular potential, though they do not have a complete $SU(1,1)$ algebra for the non-hermite pseudo-Hamiltonian. By exploring the un-deformed Hamiltonian with known radial raising and lowering operators (Newmarch and Golding, 1978; Chen *et al.*, 2000), it is expected that a complete $SU(1,1)$ algebra may arise.

At last, the similarity between radial problem and one-dimensional potentials encourages us to deal with the radial problem in the same way. And the corresponding algebra turns to approach $SU(1,1)$ algebra but for $J_0 \neq J_0^\dagger, J_+^\dagger \neq J_-$.

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